01. Quantitative Aptitude - Comprehensive Textbook Reference

> **Module Focus**: Exhaustive Quantitative Problem Solving, Textbook-Level Derivations, Speed Math Shortcuts, Modular Arithmetic, and Step-by-Step Worked Numerical Examples.

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1. Number Systems, Divisibility & Remainder Theorems

1.1 Classification of Numbers & Base Systems

- **Real Numbers ($\mathbb{R}$)**: Combination of Rational ($\mathbb{Q}$) and Irrational ($\mathbb{Q}'$) numbers.
- **Rational Numbers ($\mathbb{Q}$)**: Can be expressed as $\frac{p}{q}$, where $p, q \in \mathbb{Z}$ and $q \neq 0$.
- **Prime Numbers**: Integers > 1 with exactly two distinct positive divisors: 1 and itself.
- *Note*: 2 is the only even prime number. Prime numbers > 3 can be expressed in the form $6k \pm 1$ (though not all numbers of $6k \pm 1$ form are prime).
- **Coprime Numbers**: Two numbers a and b are coprime if $\gcd(a, b) = 1$.

1.2 Divisibility Rules & Rigorous Proofs

Divisibility Rules Summary Table:

Divisor	Condition
2	Last digit is even ($0, 2, 4, 6, 8$).
3	Sum of all digits is divisible by 3.
4	Number formed by the last two digits is divisible by 4.

Divisor	Condition
5	Last digit is 0 or 5.
6	Number is divisible by both 2 and 3.
7	Double the last digit and subtract it from the remaining number; result is divisible by 7.
8	Number formed by the last three digits is divisible by 8.
9	Sum of all digits is divisible by 9.
11	Difference between the sum of digits at odd positions and even positions is 0 or a multiple of 11.
13	Multiply last digit by 4 and add to remaining number; result is divisible by 13.

Derivations & Proofs:

Proof of Divisibility by 3 and 9:

Let any n-digit positive integer N be represented in expanded decimal notation as: $$N = a_n \cdot 10^n + a_{n-1} \cdot 10^{n-1} + \dots + a_1 \cdot 10^1 + a_0$$ Observe modular congruence modulo 3 and 9: $$10 \equiv 1 \pmod 3 \implies 10^k \equiv 1^k \equiv 1 \pmod 3$$ $$10 \equiv 1 \pmod 9 \implies 10^k \equiv 1^k \equiv 1 \pmod 9$$ Substituting $10^k \equiv 1 \pmod m$ (where $m = 3$ or 9): $$N = \sum_{k=0}^n a_k \cdot 10^k \equiv \sum_{k=0}^n a_k \cdot (1)^k = \sum_{k=0}^n a_k \pmod m$$ Thus, $N \pmod m = 0 \iff \sum a_k \pmod m = 0$. $\blacksquare$

Proof of Divisibility by 11:

Using base-10 expansion modulo 11: $$10 \equiv -1 \pmod{11} \implies 10^k \equiv (-1)^k \pmod{11}$$ Substituting into N: $$N = \sum_{k=0}^n a_k \cdot 10^k \equiv \sum_{k=0}^n a_k \cdot (-1)^k \pmod{11}$$ $$N \equiv (a_0 + a_2 + a_4 + \dots) - (a_1 + a_3 + a_5 + \dots) \pmod{11}$$ Hence, N is divisible by 11 if and only if $(\text{Sum}_{\text{odd pos}} - \text{Sum}_{\text{even pos}})$ is a multiple of 11. $\blacksquare$

1.3 Prime Factors, HCF & LCM Properties

For any integer $N > 1$ with prime factorization: $$N = p_1^{a_1} \cdot p_2^{a_2} \cdot p_3^{a_3} \cdots p_k^{a_k}$$

1. **Total Number of Factors $D(N)$**: $$D(N) = (a_1 + 1)(a_2 + 1)(a_3 + 1)\dots(a_k + 1)$$
2. **Sum of Factors $S(N)$**: $$S(N) = \left(\frac{p_1^{a_1+1} - 1}{p_1 - 1}\right) \left(\frac{p_2^{a_2+1} - 1}{p_2 - 1}\right) \dots \left(\frac{p_k^{a_k+1} - 1}{p_k - 1}\right)$$
3. **Product of Factors $P(N)$**: $$P(N) = N^{\frac{D(N)}{2}}$$

Fundamental HCF & LCM Identities:

- **Product Rule**: For any two positive integers a and b: $$\text{HCF}(a, b) \times \text{LCM}(a, b) = a \times b$$

- **Fractions**: $$\text{HCF}\left(\frac{a}{b}, \frac{c}{d}\right) = \frac{\text{HCF}(a, c)}{\text{LCM}(b, d)}, \quad \text{LCM}\left(\frac{a}{b}, \frac{c}{d}\right) = \frac{\text{LCM}(a, c)}{\text{HCF}(b, d)}$$

1.4 Trailing Zeroes & Highest Power of Prime ($n!$)

Legendre's Formula:

The exponent of a prime number p in the prime factorization of $n!$, denoted $E_p(n!)$, is given by: $$E_p(n!) = \lfloor \frac{n}{p} \rfloor + \lfloor \frac{n}{p^2} \rfloor + \lfloor \frac{n}{p^3} \rfloor + \dots = \sum_{k=1}^{\infty} \lfloor \frac{n}{p^k} \rfloor$$

Trailing Zeroes in $n!$:

Trailing zeroes are produced by factors of $10 = 2 \times 5$. Since prime factor 2 is always more abundant than 5 in $n!$, the number of trailing zeroes equals $E_5(n!)$: $$\text{Trailing Zeroes in } n! = \lfloor \frac{n}{5} \rfloor + \lfloor \frac{n}{25} \rfloor + \lfloor \frac{n}{125} \rfloor + \dots$$

1.5 Remainder Theorems & Modular Arithmetic

1. Fermat's Little Theorem:

If p is a prime number and a is an integer such that $\gcd(a, p) = 1$: $$a^{p-1} \equiv 1 \pmod p$$

2. Euler's Totient Function & Euler's Theorem:

Euler's totient function $\phi(n)$ counts positive integers up to n that are coprime to n: $$\phi(n) = n \left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right)\dots\left(1 - \frac{1}{p_k}\right)$$ For any integer a coprime to n ($\gcd(a, n) = 1$): $$a^{\phi(n)} \equiv 1 \pmod n$$

3. Wilson's Theorem:

If p is a prime number: $$(p - 1)! \equiv -1 \pmod p \quad \text{or} \quad (p - 1)! + 1 \equiv 0 \pmod p$$

4. Cyclicity of Unit Digits:

To find the unit digit of a^b: 1. Compute $r = b \pmod 4$. If $r = 0$, set $r = 4$. 2. Unit digit of $a^b = \text{Unit digit of } (a \pmod{10})^r$.

Base Last Digit	Cyclicity Sequence	Cycle Length
0, 1, 5, 6	Constant ($0, 1, 5, 6$)	1
4	$4, 6$ (Odd exponent $\to 4$, Even exponent $\to 6$)	2
9	$9, 1$ (Odd exponent $\to 9$, Even exponent $\to 1$)	2
2	$2, 4, 8, 6$	4

Base Last Digit	Cyclicity Sequence	Cycle Length
3	$3, 9, 7, 1$	4
7	$7, 9, 3, 1$	4
8	$8, 4, 2, 6$	4

1.6 Worked Numerical Examples

Worked Example 1.1 (Factors & Trailing Zeroes):

Problem: Find (i) the total number of factors of 360, and (ii) the number of trailing zeroes in $250!$.

Solution: 1. Prime factorization of 360: $$360 = 2^3 \times 3^2 \times 5^1$$ - Total number of factors $D(360) = (3+1)(2+1)(1+1) = 4 \times 3 \times 2 = \mathbf{24}$. 2. Trailing zeroes in $250!$: $$E_5(250!) = \lfloor \frac{250}{5} \rfloor + \lfloor \frac{250}{25} \rfloor + \lfloor \frac{250}{125} \rfloor + \lfloor \frac{250}{625} \rfloor$$ $$E_5(250!) = 50 + 10 + 2 + 0 = \mathbf{62} \text{ zeroes}.$$

Worked Example 1.2 (Fermat's Theorem & Cyclicity):

Problem: Find the remainder when 2^{100} is divided by 101.

Solution: - Note that $p = 101$ is a prime number, and $\gcd(2, 101) = 1$. - Applying Fermat's Little Theorem ($a^{p-1} \equiv 1 \pmod p$): $$2^{101-1} = 2^{100} \equiv 1 \pmod{101}$$ - Thus, the remainder is $ \mathbf{1}$.

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2. Time & Work / Pipes & Cisterns

2.1 Core Mathematical Model & Efficiency Concept

Work done (W) is directly proportional to efficiency (E) and time (T): $$W = E \times T$$

If a worker completes total work $W = 1$ unit in T days: $$\text{Efficiency } (E) = \frac{1}{T} \text{ (work done per day)}$$

Two-Worker Combined Time Derivation:

Let Worker A take A days and Worker B take B days to complete a work independently. - Rate of A $= E_A = \frac{1}{A}$ - Rate of B $= E_B = \frac{1}{B}$ - Combined Rate $= E_{A+B} = E_A + E_B = \frac{1}{A} + \frac{1}{B} = \frac{A + B}{A \cdot B}$

Therefore, the combined time T_{A+B} to complete 1 unit of work is: $$T_{A+B} = \frac{1}{E_{A+B}} = \frac{A \cdot B}{A + B}$$

Three-Worker Combined Time Formula:

$$T_{A+B+C} = \frac{A \cdot B \cdot C}{AB + BC + CA}$$

2.2 The Unit Work (LCM) Method

Instead of fractions, set Total Work $W = \text{LCM}(A, B, C, \dots)$. 1. Compute individual efficiencies: $E_i = \frac{W}{T_i}$. 2. Combine efficiencies according to working schedule. 3. Total Time required $= \frac{\text{Total Work}}{\text{Net Efficiency}}$.

2.3 Man-Days & Chain Rule Formula

For workforce scaling problems where work W_1 requires M_1 men working H_1 hours/day for D_1 days with efficiency E_1, and W_2 requires M_2 men working H_2 hours/day for D_2 days with efficiency E_2:

$$\frac{M_1 \cdot D_1 \cdot H_1 \cdot E_1}{W_1} = \frac{M_2 \cdot D_2 \cdot H_2 \cdot E_2}{W_2}$$

2.4 Division of Wages

Wages are distributed strictly proportional to the amount of work completed by each individual. If all workers work for the same duration: $$\text{Wage ratio} = E_A : E_B : E_C = \frac{1}{T_A} : \frac{1}{T_B} : \frac{1}{T_C}$$

2.5 Pipes & Cisterns Mechanics

- **Inlet Pipe**: Fills the tank (Positive efficiency: $+E = +\frac{1}{T_{\text{fill}}}$).
- **Outlet Pipe / Leak**: Empties the tank (Negative efficiency: $-E = -\frac{1}{T_{\text{empty}}}$).
- **Net Filling Rate**: $$E_{\text{net}} = \sum E_{\text{inlets}} - \sum E_{\text{outlets}}$$

2.6 Worked Numerical Examples

Worked Example 2.1 (Alternate Working Days):

Problem: A can complete a job in 12 days, and B can complete it in 18 days. If they work on alternate days starting with A, in how many days will the work be completed?

Solution: 1. Take Total Work $W = \text{LCM}(12, 18) = 36 \text{ units}$. 2. Calculate efficiencies: - $E_A = \frac{36}{12} = 3 \text{ units/day}$ - $E_B = \frac{36}{18} = 2 \text{ units/day}$ 3. Two-day cycle work (Day 1: A, Day 2: B): $$\text{Work in 1 cycle (2 days)} = E_A + E_B = 3 + 2 = 5 \text{ units}$$ 4. Number of complete cycles to reach near 36 units: $$\text{Cycles} = \lfloor \frac{36}{5} \rfloor = 7 \text{ cycles} \implies 7 \times 2 = 14 \text{ days}$$ $$\text{Work completed in 14 days} = 7 \times 5 = 35 \text{ units}$$ 5. Remaining work $= 36 - 35 = 1 \text{ unit}$. 6. On Day 15, it is A's turn ($E_A = 3 \text{ units/day}$): $$\text{Time taken by A for 1 unit} = \frac{1}{3} \text{ days}$$ 7. Total time $= 14 + \frac{1}{3} = \mathbf{14 \frac{1}{3} \text{ days}}$.

Worked Example 2.2 (Pipes with Leak):

Problem: Pipe A can fill a cistern in 8 hours, but due to a leak at the bottom, it takes 10 hours to fill. How long will the leak take to empty a full cistern alone?

Solution: 1. Let capacity of cistern $= \text{LCM}(8, 10) = 40 \text{ units}$. 2. Rate of Pipe A ($E_A$) $= \frac{40}{8} = +5 \text{ units/hour}$. 3. Net rate of Pipe A + Leak ($E_A - E_L$) $= \frac{40}{10} = +4 \text{ units/hour}$. 4. Efficiency of Leak (E_L): $$5 - E_L = 4 \implies E_L = 1 \text{ unit/hour}$$ 5. Time taken by Leak to empty full tank $= \frac{40}{1} = \mathbf{40 \text{ hours}}$.

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3. Speed, Distance & Time / Trains / Boats & Streams

3.1 Fundamental Equation & Unit Conversions

$$S = \frac{D}{T} \implies D = S \times T, \quad T = \frac{D}{S}$$

Unit Conversion Derivation:

$$1 \text{ km/h} = \frac{1000 \text{ meters}}{3600 \text{ seconds}} = \frac{5}{18} \text{ m/s}$$ $$1 \text{ m/s} = \frac{\frac{1}{1000} \text{ km}}{\frac{1}{3600} \text{ hours}} = \frac{3600}{1000} \text{ km/h} = \frac{18}{5} \text{ km/h}$$

3.2 Average Speed Derivations

1. Equal Distances Case (Harmonic Mean):

Suppose a body travels distance d at speed S_1 and another equal distance d at speed S_2. - $T_1 = \frac{d}{S_1}, \quad T_2 = \frac{d}{S_2}$ - $T_{\text{total}} = T_1 + T_2 = d \left(\frac{1}{S_1} + \frac{1}{S_2}\right) = d \left(\frac{S_1 + S_2}{S_1 S_2}\right)$ - $D_{\text{total}} = 2d$

$$\text{Average Speed } (S_{\text{avg}}) = \frac{D_{\text{total}}}{T_{\text{total}}} = \frac{2d}{d \left(\frac{S_1 + S_2}{S_1 S_2}\right)} = \frac{2 S_1 S_2}{S_1 + S_2}$$

2. Equal Time Intervals Case (Arithmetic Mean):

If a body travels for time t at speed S_1 and for time t at speed S_2: $$S_{\text{avg}} = \frac{S_1 \cdot t + S_2 \cdot t}{2t} = \frac{S_1 + S_2}{2}$$

3.3 Relative Speed Principles

When two bodies move relative to each other: - **Same Direction**: $S_{\text{rel}} = |S_1 - S_2|$ - **Opposite Direction**: $S_{\text{rel}} = S_1 + S_2$

3.4 Problems on Trains

Let Train length $= L_T$, Speed $= S_T$.

1. **Crossing a Point Object** (Pole, Stationary Man, Signal Tree of length ≈ 0): $$D = L_T \implies T = \frac{L_T}{S_T}$$

2. **Crossing a Platform / Bridge / Tunnel** of length L_P: $$D = L_T + L_P \implies T = \frac{L_T + L_P}{S_T}$$
3. **Two Trains Crossing Each Other** (Lengths L_1, L_2, Speeds S_1, S_2): $$D_{\text{total}} = L_1 + L_2$$ $$T = \frac{L_1 + L_2}{S_1 + S_2} \quad (\text{Opposite Direction})$$ $$T = \frac{L_1 + L_2}{|S_1 - S_2|} \quad (\text{Same Direction})$$

3.5 Boats and Streams Mechanics

Let Speed of Boat in Still Water $= b$, Speed of Current/Stream $= s$.

- **Downstream Speed (U)**: Motion in direction of current. $$U = b + s$$
- **Upstream Speed (V)**: Motion against direction of current. $$V = b - s$$

Derivation of b and s:

Adding U and V: $$U + V = (b + s) + (b - s) = 2b \implies b = \frac{U + V}{2}$$ Subtracting V from U: $$U - V = (b + s) - (b - s) = 2s \implies s = \frac{U - V}{2}$$

3.6 Worked Numerical Examples

Worked Example 3.1 (Train Crossing Platform and Man):

Problem: A train moving at uniform speed crosses a 200 m long platform in 20 seconds and a electric pole in 8 seconds. Find the length and speed of the train.

Solution: 1. Let length of train be L meters and speed be S m/s. 2. Crossing pole equation: $$S = \frac{L}{8} \implies L = 8S$$ 3. Crossing platform equation: $$S = \frac{L + 200}{20}$$ 4. Substitute $L = 8S$: $$S = \frac{8S + 200}{20} \implies 20S = 8S + 200 \implies 12S = 200 \implies S = \frac{50}{3} \text{ m/s}$$ 5. Speed in km/h $= \frac{50}{3} \times \frac{18}{5} = \mathbf{60 \text{ km/h}}$. 6. Length of train $L = 8 \times \frac{50}{3} = \mathbf{\frac{400}{3} = 133.33 \text{ meters}}$.

Worked Example 3.2 (Boats & Streams):

Problem: A boat travels 24 km upstream and 28 km downstream in 6 hours. It can also travel 30 km upstream and 21 km downstream in 6.5 hours. Find the speed of the boat in still water and speed of current.

Solution: 1. Let $\frac{1}{V} = x$ (where $V = b-s$) and $\frac{1}{U} = y$ (where $U = b+s$). 2. Form system of equations: $$24x + 28y = 6 \implies 12x + 14y = 3 \quad \text{--- (1)}$$ $$30x + 21y = 6.5 \implies 60x + 42y = 13 \quad \text{--- (2)}$$ 3. Multiply (1) by 3: $$36x + 42y = 9 \quad \text{--- (3)}$$ 4. Subtract (3) from (2): $$24x = 4 \implies x = \frac{1}{6} \implies V = 6 \text{ km/h}$$ 5. Substitute $x = 1/6$ into (1): $$12\left(\frac{1}{6}\right) + 14y = 3 \implies 2 + 14y = 3 \implies 14y = 1 \implies y = \frac{1}{14} \implies U = 14 \text{ km/h}$$ 6. Solve for b and s: $$b = \frac{U + V}{2} = \frac{14 + 6}{2} = \mathbf{10 \text{ km/h}}$$ $$s = \frac{U - V}{2} = \frac{14 - 6}{2} = \mathbf{4 \text{ km/h}}$$

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4. Profit, Loss & Discount

4.1 Fundamental Definitions & Core Equations

- **Cost Price (CP)**: Total expenditure incurred to acquire/produce item.
- **Selling Price (SP)**: Price at which item is sold.
- **Marked Price (MP)**: List price tagged on item.

$$\text{Profit } (P) = \text{SP} - \text{CP} \quad (\text{if } \text{SP} > \text{CP})$$ $$\text{Loss } (L) = \text{CP} - \text{SP} \quad (\text{if } \text{CP} > \text{SP})$$ $$\text{Profit \%} = \left(\frac{\text{SP} - \text{CP}}{\text{CP}} \right) \times 100$$ $$\text{Loss \%} = \left(\frac{\text{CP} - \text{SP}}{\text{CP}}\right) \times 100$$ $$\text{SP} = \text{CP} \times \left(1 + \frac{P\%}{100}\right) = \text{MP} \times \left(1 - \frac{D\%}{100}\right)$$

4.2 Markup & Successive Discount Formula Derivation

Successive Discount Derivation:

Let marked price be M. If two successive discounts $d_1\%$ and $d_2\%$ are applied: 1. Price after first discount $= M \left(1 - \frac{d_1}{100}\right)$. 2. Price after second discount $= M \left(1 - \frac{d_1}{100}\right) \left(1 - \frac{d_2}{100}\right) = M \left(1 - \left[\frac{d_1 + d_2 - \frac{d_1 d_2}{100}}{100}\right]\right)$.

Thus, single equivalent discount $D_{\text{eq}}\%$ is: $$D_{\text{eq}} = d_1 + d_2 - \frac{d_1 \cdot d_2}{100}$$

4.3 Selling Two Articles at Same SP with Same Gain/Loss %

Derivation:

Suppose two articles are sold at the SAME selling price S. One at a profit of $x\%$ and the other at a loss of $x\%$. 1. Cost price of first article $\text{CP}_1 = \frac{S}{1 + x/100} = \frac{100 S}{100 + x}$. 2. Cost price of second article $\text{CP}_2 = \frac{S}{1 - x/100} = \frac{100 S}{100 - x}$. 3. Total Cost Price: $$\text{CP}_{\text{total}} = 100 S \left(\frac{1}{100 + x} + \frac{1}{100 - x} \right) = 100 S \left(\frac{200}{10000 - x^2} \right) = \frac{20000 S}{10000 - x^2}$$ 4. Total Selling Price $\text{SP}_{\text{total}} = 2S$. 5. Loss $= \text{CP}_{\text{total}} - \text{SP}_{\text{total}} = 2S \left(\frac{10000}{10000 - x^2} - 1 \right) = 2S \left(\frac{x^2}{10000 - x^2} \right)$. 6. Net Percentage Loss: $$\text{Net Loss \%} = \left(\frac{\text{Loss}}{\text{CP}_{\text{total}}}\right) \times 100 = \left(\frac{2S x^2 / (10000 - x^2)}{20000 S / (10000 - x^2)} \right) \times 100 = \frac{x^2}{100} = \left(\frac{x}{10}\right)^2$$

Hence, there is ALWAYS a net loss of: $$\mathbf{\text{Net Loss \%} = \left(\frac{x}{10}\right)^2}$$

4.4 Faulty Weights & Dishonest Shopkeepers

If a shopkeeper claims to sell at cost price but uses a false weight W_{false} instead of true weight W_{true}:

$$\text{Gain \%} = \left(\frac{\text{True Weight} - \text{False Weight}}{\text{False Weight}} \right) \times 100 = \left(\frac{\text{Error}}{\text{True Value} - \text{Error}} \right) \times 100$$

4.5 Worked Numerical Examples

Worked Example 4.1 (Faulty Weights):

Problem: A dishonest trader marks his goods at 20% above CP and allows a discount of 10%. Furthermore, he uses a 800 g weight instead of a 1 kg weight. Find his total profit percentage.

Solution: 1. Let actual CP of $1000 \text{ g} = ₹1000 \implies \text{CP per gram} = ₹1$. 2. Marked price for $1000 \text{ g} = 1000 \times 1.20 = ₹1200$. 3. Selling price after 10% discount $= 1200 \times 0.90 = ₹1080$ for 1 kg claimed. 4. But trader actually delivers only 800 g. 5. Trader's actual CP for $800 \text{ g} = ₹800$. 6. Actual Profit $= \text{SP} - \text{Actual CP} = ₹1080 - ₹800 = ₹280$. 7. Net Profit $\% = \frac{280}{800} \times 100 = \mathbf{35\%}$.

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5. Simple Interest (SI) & Compound Interest (CI)

5.1 Simple Interest Theory & Derivation

Simple interest is calculated uniformly on principal P over time T years at annual rate $R\%$:

$$I_{\text{SI}} = \frac{P \cdot R \cdot T}{100}$$ $$\text{Total Amount } (A) = P + I_{\text{SI}} = P \left(1 + \frac{R \cdot T}{100}\right)$$

5.2 Compound Interest Theory & Derivation

Derivation from Annual Compounding:

- Year 1 Amount: $A_1 = P\left(1 + \frac{R}{100}\right)$
- Year 2 Amount: $A_2 = A_1\left(1 + \frac{R}{100}\right) = P\left(1 + \frac{R}{100}\right)^2$
- By mathematical induction, for n years: $$A = P\left(1 + \frac{R}{100}\right)^n$$ $$I_{\text{CI}} = A - P = P\left[\left(1 + \frac{R}{100}\right)^n - 1\right]$$

Compounding Periods Table:

Compounding Frequency	Effective Rate (R')	Effective Periods (n')	Formula
Annual	R	n	$A = P\left(1 + \frac{R}{100}\right)^n$
Half-Yearly (Semi-Annual)	$\frac{R}{2}$	$2n$	$A = P\left(1 + \frac{R}{200}\right)^{2n}$
Quarterly	$\frac{R}{4}$	$4n$	$A = P\left(1 + \frac{R}{400}\right)^{4n}$

5.3 CI vs SI Difference Formulas Derivation

2-Year Difference (D_2):

$$I_{\text{SI}}^{(2)} = \frac{2 P R}{100}$$ $$I_{\text{CI}}^{(2)} = P\left(1 + \frac{R}{100}\right)^2 - P = P\left(1 + \frac{2R}{100} + \frac{R^2}{10000}\right) - P = \frac{2 P R}{100} + P\left(\frac{R}{100}\right)^2$$ $$D_2 = I_{\text{CI}}^{(2)} - I_{\text{SI}}^{(2)} = \mathbf{P\left(\frac{R}{100}\right)^2}$$

3-Year Difference (D_3):

$$D_3 = I_{\text{CI}}^{(3)} - I_{\text{SI}}^{(3)} = \mathbf{P\left(\frac{R}{100}\right)^2 \left(3 + \frac{R}{100}\right)}$$

5.4 Equal Annual Installment Formulas

1. Installments under Simple Interest:

If debt D is cleared in n equal annual installments of x at $R\%$ SI per annum: $$D = n \cdot x + \frac{R \cdot x}{100} \times \frac{(n-1)n}{2}$$

2. Installments under Compound Interest:

If principal P is repaid in n equal annual installments of x at $R\%$ CI per annum: $$P = \frac{x}{\left(1 + \frac{R}{100}\right)^1} + \frac{x}{\left(1 + \frac{R}{100}\right)^2} + \dots + \frac{x}{\left(1 + \frac{R}{100}\right)^n}$$

5.5 Worked Numerical Examples

Worked Example 5.1 (CI & SI Difference):

Problem: The difference between compound interest and simple interest on a certain sum of money for 2 years at 8% per annum is $₹64$. Find the principal sum.

Solution: 1. Use 2-year difference formula: $$D_2 = P\left(\frac{R}{100}\right)^2$$ 2. Substitute values: $$64 = P\left(\frac{8}{100}\right)^2 = P \left(\frac{64}{10000}\right)$$ 3. Solve for P: $$P = \frac{64 \times 10000}{64} = \mathbf{₹10,000}$. #### Worked Example 5.2 (CI Installments): **Problem**: A sum of $₹13,360$ is borrowed at $8 \frac{3}{4}\%$ per annum compound interest and paid back in two equal annual installments. Find the value of each installment. **Solution**: 1. Interest rate $R = 8.75\% = \frac{35}{4}\% = \frac{7}{80}$. 2. Compounding factor $1 + \frac{R}{100} = 1 + \frac{7}{80} = \frac{87}{80}$. 3. Let each installment be x. Applying CI installment formula: $$13360 = \frac{x}{\frac{87}{80}} + \frac{x}{\left(\frac{87}{80}\right)^2} = \frac{80x}{87} + \frac{6400x}{7569}$$ 4. Take common denominator 7569: $$13360 = \frac{80 \times 87 x + 6400 x}{7569} = \frac{6960x + 6400x}{7569} = \frac{13360 x}{7569}$$ 5. Solve for x: $$x = \frac{13360 \times 7569}{13360} = \mathbf{₹7,569}$.

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6. Permutations, Combinations & Probability

6.1 Fundamental Counting Principles

- **Addition Principle (OR Rule)**: If event A can occur in m ways and event B in n mutually exclusive ways, then A or B can occur in $m + n$ ways.
- **Multiplication Principle (AND Rule)**: If event A can occur in m ways and event B in n independent ways, then A and B together can occur in $m \times n$ ways.

6.2 Permutations (nPr) Derivation & Special Cases

Mathematical Derivation:

Number of ways to fill r distinct positions from n distinct objects: - Position 1: n choices - Position 2: $n-1$ choices - Position r: $n - (r - 1) = n - r + 1$ choices

$$nPr = n(n-1)(n-2)\dots(n-r+1) = \frac{n(n-1)\dots(n-r+1)(n-r)!}{(n-r)!} = \frac{n!}{(n-r)!}$$

Special Permutation Rules:

1. **Permutations with Repetition**: Arrangements of n items where p are of type 1, q of type 2, r of type 3: $$\text{Ways} = \frac{n!}{p! \cdot q! \cdot r!}$$
2. **Circular Permutations**:
3. Distinct directional arrangements around a circle: $(n - 1)!$
4. Non-directional (Necklaces / Garland of flowers where clockwise and counter-clockwise are identical): $\frac{(n - 1)!}{2}$

6.3 Combinations (nCr) Derivation & Geometric Applications

Mathematical Derivation:

Since selection order does not matter in combinations, each combination of r objects can be arranged internally in $r!$ ways. $$nPr = nCr \times r! \implies nCr = \frac{nPr}{r!} = \frac{n!}{r!(n-r)!}$$

Key Combinatorial Identities:

- $nCr = nC_{n-r}$
- Pascal's Identity: $nCr + nC_{r-1} = ^{(n+1)}C_r$
- Sum of Combinations: $\sum_{k=0}^n nCk = 2^n$

Geometric Formulas:

- Number of straight lines from n non-collinear points $= nC_2$
- Number of triangles from n non-collinear points $= nC_3$
- Number of diagonals in an n-sided convex polygon: $$\text{Diagonals} = nC_2 - n = \frac{n(n-1)}{2} - n = \frac{n(n-3)}{2}$$

6.4 Derangements (D_n) Formula

A derangement is a permutation of n items such that no item appears in its original position.

$$D_n = n! \sum_{k=0}^n \frac{(-1)^k}{k!} = n! \left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \dots + \frac{(-1)^n}{n!} \right)$$

Values: $D_1 = 0$, $D_2 = 1$, $D_3 = 2$, $D_4 = 9$, $D_5 = 44$.

6.5 Probability Axioms & Conditional Probability

For sample space S and event A: $$0 \le P(A) \le 1, \quad P(S) = 1, \quad P(A') = 1 - P(A)$$

Inclusion-Exclusion Principle:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Conditional Probability & Bayes' Theorem:

$$P(A | B) = \frac{P(A \cap B)}{P(B)} \quad (\text{for } P(B) > 0)$$

Bayes' Theorem:

For mutually exclusive and exhaustive events $E_1, E_2, \dots, E_n$: $$P(E_k | A) = \frac{P(E_k) \cdot P(A | E_k)}{\sum_{i=1}^n P(E_i) \cdot P(A | E_i)}$$

6.6 Worked Numerical Examples

Worked Example 6.1 (Permutations with Restrictions):

Problem: In how many ways can the letters of the word **'TRIANGLE'** be arranged such that all vowels always come together?

Solution: 1. Word: 'TRIANGLE' (Total 8 letters: T, R, I, A, N, G, L, E). 2. Vowels: $\{I, A, E\}$ (3 vowels). Consonants: $\{T, R, N, G, L\}$ (5 consonants). 3. Treat the 3 vowels as a single block $[IAE]$. 4. Total units to arrange $= 5 \text{ consonants} + 1 \text{ block} = 6 \text{ units}$. 5. Number of ways to arrange 6 units $= 6! = 720$. 6. Internal arrangements of 3 vowels inside the block $= 3! = 6$. 7. Total arrangements $= 6! \times 3! = 720 \times 6 = \mathbf{4,320}$.

Worked Example 6.2 (Bayes' Theorem Application):

Problem: Bag I contains 3 Red and 4 Black balls. Bag II contains 5 Red and 3 Black balls. One bag is selected at random and a ball is drawn from it. If the drawn ball is Red, find the probability that it was drawn from Bag I.

Solution: 1. Let E_1 = Selecting Bag I, E_2 = Selecting Bag II, A = Drawing a Red ball. 2. Prior probabilities: $P(E_1) = \frac{1}{2}, P(E_2) = \frac{1}{2}$. 3. Conditional probabilities of Red ball: $$P(A | E_1) = \frac{3}{3+4} = \frac{3}{7}$$ $$P(A | E_2) = \frac{5}{5+3} = \frac{5}{8}$$ 4. Apply Bayes' Theorem for $P(E_1 | A)

$: $$P(E_1 | A) = \frac{P(E_1) \cdot P(A | E_1)}{P(E_1) \cdot P(A | E_1) + P(E_2) \cdot P(A | E_2)}$$ $$P(E_1 | A) = \frac{\frac{1}{2} \times \frac{3}{7}}{\left(\frac{1}{2} \times \frac{3}{7}\right) + \left(\frac{1}{2} \times \frac{5}{8}\right)} = \frac{\frac{3}{7}}{\frac{3}{7} + \frac{5}{8}} = \frac{\frac{3}{7}}{\frac{24 + 35}{56}} = \frac{\frac{3}{7}}{\frac{59}{56}} = \frac{3}{7} \times \frac{56}{59} = \mathbf{\frac{24}{59}}$. --- ## 7. CoCubes High-Frequency Quants Question Patterns, Speed Shortcuts & Solved PYQs > **Source Research**: Extracted from PrepInsta, FACE Prep, IndiaBIX, and GeeksforGeeks CoCubes placement archives. > **Section Overview**: 15–20 Questions in 15–20 Minutes (Moderate difficulty, non-adaptive/adaptive modules, 0 negative marking). ### 7.1 High-Frequency Question Patterns & Speed Tricks Table | Topic Cluster | Core CoCubes Question Pattern | Speed Shortcut Formula / Technique | Target Time | | :--- | :--- | :--- | :--- | | **Time, Speed & Distance** | Trains starting at different times meeting at a point | $\text{Distance to meet} = \text{Speed}_2 \times \frac{\text{Initial Lead Distance}}{\text{Relative Speed}}$ | $< 45 \text{ sec}$ | | **Time & Work** | M_1 workers finish in D_1 days; find workers needed for D_2 days | Chain Rule: $M_1 \times D_1 = M_2 \times D_2 \implies M_2 = \frac{M_1 D_1}{D_2}$ | $< 30 \text{ sec}$ | | **Ratio & Proportion** | Given $a:b$ and $b:c$, find $a:c$ or combine $a:b:c$ | $\frac{a}{c} = \frac{a}{b} \times \frac{b}{c}$; For $a:b:c$, equalize b's value using LCM | $< 30 \text{ sec}$ | | **Algebra & Conditionals** | "If $A = \frac{2}{3}B, B = \frac{2}{3}C, C = \frac{2}{3}D$, find B in terms of D" | Direct Back-Substitution: $B = \frac{2}{3} \times \left(\frac{2}{3}D\right) = \frac{4}{9}D$ | $< 25 \text{ sec}$ | | **Profit & Loss** | Two successive discounts $d_1\%$ and $d_2\%$ on Marked Price | $D_{\text{effective}} = d_1 + d_2 - \frac{d_1 \times d_2}{100}\%$ | $< 30 \text{ sec}$ | | **Number Systems** | Remainder of $a^n \div m$ where $a = m \pm 1$ | $(m \pm 1)^n \equiv (\pm 1)^n \pmod m$ | $< 20 \text{ sec}$ | --- ### 7.2 Solved CoCubes Quantitative Aptitude PYQs #### PYQ 7.1 (Time, Speed & Distance - Train Meeting Time) **Problem**: Two trains leave a railway station at 10:00 a.m. and 10:30 a.m., traveling in the same direction at speeds of 60 km/h and 75 km/h respectively. At what distance from the starting station will the two trains meet? **Options**: A) 120 km B) 150 km C) 180 km D) 200 km **Step-by-Step Solution & Speed Trick**: 1. **Initial Lead of Train 1**: Train 1 travels alone from 10:00 a.m. to 10:30 a.m. (0.5 hours). $$\text{Lead Distance} = 60 \text{ km/h} \times 0.5 \text{ h} = 30 \text{ km}$$ 2. **Relative Speed**: Since both trains move in the same direction: $$\text{Relative Speed} = 75 - 60 = 15 \text{ km/h}$$ 3. **Time to Catch Up**: $$\text{Time to meet} = \frac{\text{Lead Distance}}{\text{Relative Speed}} = \frac{30 \text{ km}}{15 \text{ km/h}} = 2 \text{ hours}$$ 4. **Distance from Station**: Distance covered by Train 2 in 2 hours: $$\text{Distance} = 75 \text{ km/h} \times 2 \text{ h} = \mathbf{150 \text{ km}}$$ **Correct Answer**: **`B) 150 km`** --- #### PYQ 7.2 (Time & Work - Man-Days Inverse Relation) **Problem**: 28 children can complete a piece of work in 50 days. How many children are required to complete the exact same work in 35 days? **Options**: A) 36 B) 40 C) 42 D) 45 **Step-by-Step Solution & Speed Trick**: 1. Apply the Man-Days formula $M_1 \times D_1 = M_2 \times D_2$: $$28 \times 50 = M_2 \times 35$$ 2. Solve for M_2: $$M_2 = \frac{28 \times 50}{35} = \frac{4 \times 50}{5} = 4 \times 10 = \mathbf{40 \text{ children}}$$ **Correct Answer**: **`B) 40`** --- #### PYQ 7.3 (Ratio & Proportion - Compound Ratio) **Problem**: If $a : b = 3 : 4$ and $b : c = 5 : 7$, find the ratio $a : c$. **Options**: A) $15 : 28$ B) $3 : 7$ C) $5 : 4$ D) $12 : 35$ **Step-by-Step Solution & Speed Trick**: 1. Express as fractions: $\frac{a}{b} = \frac{3}{4}$ and $\frac{b}{c} = \frac{5}{7}$. 2. Multiply the two equations to cancel b: $$\frac{a}{c} = \frac{a}{b} \times \frac{b}{c} = \frac{3}{4} \times \frac{5}{7} = \frac{15}{28}$$ 3. Thus, $a : c = \mathbf{15 : 28}$. **Correct Answer**: **`A) 15 : 28`** --- #### PYQ 7.4 (Algebraic Variable Substitution) **Problem**: If $a = \frac{2}{3}b$, $b = \frac{2}{3}c$, and $c = \frac{2}{3}d$, what fraction of d is b? **Options**: A) $\frac{2}{3}$ B) $\frac{4}{9}$ C) $\frac{8}{27}$ D) $\frac{1}{3}$ **Step-by-Step Solution & Speed Trick**: 1. Substitute $c = \frac{2}{3}d$ into the expression for b: $$b = \frac{2}{3}c = \frac{2}{3} \left(\frac{2}{3}d\right) = \mathbf{\frac{4}{9}d}$$ 2. Therefore, b is $\mathbf{\frac{4}{9}}$ of d.

Correct Answer: `B) 4/9`